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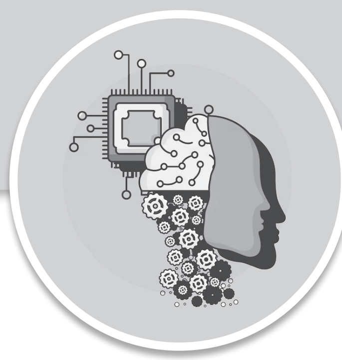
Probability and Statistics



Comprehensive Theory
with Solved Examples *and* Practice Questions

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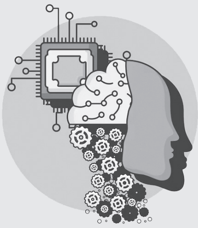

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Probability and Statistics

CHAPTER

1

Counting Techniques

1.1 INTRODUCTION

Objects (or things) can be arranged in many ways. Suppose there are three objects marked *a*, *b*, *c* on a table from these, two objects can be selected at a line in three different ways as {*a*, *b*}, {*a*, *c*}, {*b*, *c*}. In this way selection of two objects from three objects in three ways is called **Combinations**.

The above selection *ab*, *ac* can also be arranged as *ab*, *ba*, *ac*, *ca*, *bc*, *cb*. We can understand that two objects can be selected from three objects and arranged in six ways. These arrangements are called **Permutations**.

1.1.1 Fundamental Concepts

If A is a finite set, then the number of different elements in A is denoted by $n(A)$ e.g.,

If A = {2, 5, 7} then $n(A) = 3$

If C = ϕ then $n(C) = 0$

Let us assume that there are three routes say a_1, a_2, a_3 from Delhi to Noida and there are two routes, say b_1, b_2, from Noida to Agra. It may be written as:

$A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$

$n(A) = 3$; $n(B) = 2$

Now we can match the route a_1 from D to N with two routes b_1, b_2 from N, A

i.e., (a_1, b_1), (a_1, b_2)

Similarly the remaining routes can be written as

(a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2).

So to travel from D to A via N, there are 6 different routes

(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_2).

These 6 ways are nothing but the elements of the Cartesian product of the two sets A and B.

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$

$n(A \times B) = 6 = 2 \times 3 = n(A) \times n(B)$.

1.1.2 Fundamental Multiplication Principle of Counting

"Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of *m* possible out comes and if for each outcome of experiment 1 there are possible out comes of experiment 2, then together there are *mn* possible outcomes of the two experiments".

1.1.3 The Generalized basic Multiplication Principle of Counting

"If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment and if for each of the possible outcomes the first two experiments there are n_3 possible outcomes of the third experiment and so on, then there is a total of $n_1 \times n_2 \ldots n_r$ possible outcomes of the r experiments".

Keywords to distinguish permutations from combinations:

Permutations : ordered, arrangement, sequence

Combinations : unordered, selection, set

Some useful Properties from Number Theory used in Combinatorics

1. **Method for finding the number of positive divisors of a positive integer n:** If a positive integer n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \ldots.$ where p_1, p_2 etc. are distinct prime numbers, then the number of positive divisors of n is given by the formula $(n_1 + 1)(n_2 + 1)\ldots$

   For example, the number 80 can be broken as $80 = 2^4 \times 5^1$. So the number of positive divisors of 80 is given by $(4 + 1)(1 + 1) = 10$.

2. **Method for finding the number of numbers from 1 to n, which are relatively prime to n :** The number of numbers from 1 to n, which are relatively prime to n i.e., gcd $(m, n) = 1$, is given by the Euler Totient function $\phi(n)$. If n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \ldots.$ where p_1, p_2 etc. are distinct prime numbers, then $\phi(n) = \phi(p_1^{n_1})\, \phi(p_2^{n_2})\ldots$ then by using the property

$$\phi(p^k) = p^k - p^{k-1}$$

   we can find each of $\phi(p_1^{n_1}), \phi(p_2^{n_2})\ldots$ etc.

   For example, the number of numbers from 1 to n, which are relatively prime to 80 can be found as follows: Since $80 = 2^4 \times 5^1$

   The number of numbers from 1 to n, which are relatively prime to $80 = \phi(80) = \phi(2^4) \times \phi(5^1)$

   Now $\qquad \phi(2^4) = 2^4 - 2^3 = 16 - 8 = 8$

   Similarly, $\qquad \phi(5^1) = 5^1 - 5^0 = 5 - 1 = 4$

   So, $\qquad \phi(80) = 8 \times 4 = 32$

1.2 Permutations

1.2.1 Permutations With No Repetitions

When we select objects from a set consisting of n distinct objects taking each object exactly once (no repetition), and then arrange them in a straight line, this situation is called permutations with no repetition.

The formula for counting this is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \ldots (n-r+1)$$

Example 1.1 How many 2 letter passwords are there using the letters {*a*, *b*, *c*} if no letter is allowed to be used more then once?

Solution:

$$^3P_2 = \frac{3!}{(3-2)!} = 3 \times 2 = 6$$

The 6 permutations are *ab*, *ba*, *ac*, *ca*, *bc*, *cb*.

Example 1.2 How many ways can four (distinct) dolls be arranged in a straight line?

Solution:

$$^4P_4 = \frac{4!}{(4-4)!} = 4! = 4 \times 3 \times 2 \times 1 = 24 \text{ ways.}$$

Alternately more problems can be solved by box method, which is more general and more powerful. For example arranging 4 dolls can be thought of as filling 4 boxes corresponding to position of the dolls. The first box can be filled in 4 ways. Since repetition is not allowed, the second box can be filled only in 3 ways.

The third box in 2 ways and last box in only 1 way.

$\therefore$ Total arrangements $= 4 \times 3 \times 2 \times 1 = 24$ ways

1.2.2 Permutations With Unlimited Repetition

When we select an object, from a set of distinct objects, taking each object any number of times (unlimited repetition) and arrange them in a straight line, this situation is called permutation with unlimited repetition. The formula for counting this is n^r.

Example 1.3 How many 2 letter passwords can be made from {*a*, *b*, *c*}, if a letter can be used any number of times?

Solution:

$$3^2 = 9 \text{ passwords}$$

The nine permutations are : *aa*, *ab*, *ac*, *ba*, *bb*, *bc*, *ca*, *cb*, *cc*.

NOTE

For objects such as passwords & number, if nothing is mentioned regarding repetition, the default assumption is that unlimited repetition is allowed.
Using box method we make the password by filling 2 boxes. Each can be filled in 3 ways (since repetition allowed) $3 \times 3 = 9$ passwords.

Example 1.4 If there are 10 multiple choice question with four choices for each question, How many answer sheets are possible?

Solution:

5^{10} answer sheets. We can think of each of the question as a box and each of the 10 boxes can be filled is 5 ways (Choice *a*, *b*, *c* or *d* or leave it blank):

$\therefore$ The number of ways of filling up all 10 boxes is $5 \times 5 \times 5 \times \ldots$ 10 times $= 5^{10}$

The box method is very powerful for use in all permutation problems (with repetition or without repetition).

1.4.2 Multinomial Coefficients

$$(x_1 + x_2 + ... + x_t)^n = \Sigma P(n, q_1, q_2, ... q_t) = x_1^{q1}\, x_2^{q2}\, x_3^{q3} ... x_t^{qt} \quad q_1 + q_2 ... + q_t = n$$

The coefficient of the term $x_1^3\, x_2^4\, x_3^2$ in the expansion of $(x_1 + x_2 + x_3)^9$, would be exactly $P(9; 3, 4, 2) = \frac{9!}{3!4!2!}$.

$$\Sigma P(9, q_1, q_2, ... q_t) = 3^9 \quad q_1 + q_2 ... q_t = n$$

e.g. $\Sigma P(q, q_1, q_2, ... q_3) = 3^9 \quad q_1 + q_2 + q_3 + = 9$ (summation of all multinomial coefficients)

Summary

- The number of ways of arranging n objects in a circle is $(n - 1)!$ This is less than $n!$, since in a circle many of the linear permutations become indistinguishable except for rotation.
- Combinations with no Repetitions $\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$
- Combinations with Unlimited Repetitions ${}^{n-1+r}C_r$ which can also be written as ${}^{n-1+r}C_{n-1}$.
- Distribution of Distinct Objects: $\frac{n!}{n_1! n_2! ... n_r!}$
- **Vandermondes indentity:** ${}^{n+m}C_r = {}^nC_0\,{}^mC_r + {}^nC_1\,{}^mC_{r-1} + {}^nC_2\,{}^mC_{r-2} ... {}^nC_r\,{}^mC_0$
- Special case of Vandermonde's identity is obtained by putting $r = n$
  ${}^{n+m}C_n = {}^nC_0\,{}^mC_n + {}^nC_1\,{}^mC_{n-1} + ... {}^nC_n\,{}^mC_0$
- Ball in the box problems, when balls are indistinguishable, along with upper constraints on the no of balls that can be put into a box, can be effectively converted into a problem of finding coefficient of some power of X in the expansion of a corresponding generating function.
- Solving linear inhomogeneous equations by characteristic roots method:

RHS	Form of trial particular solution (a^p_n)	
Constant	C	d
Linear	$C_0 + C_1 n$	$d_0 + d_1 n$
Quadratic	$C_0 + C_1 n + C_2 n^2$	$d_0 + d_1 n + d_2 n^2$
Power fn	$C.a^n$	da^n
Power fn * Poly	$a^n (C_0 + C_1 n ...)$	$a^n (d_0 + d_1 n ...)$

Student's Assignment

Q.1 How many strings are there of lowercase letters of length four or less?

Q.2 A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
(a) How many socks must he take out to be sure that he has at least two socks of the same color?
(b) How many socks must he take out to be sure that he has at least two black socks?

Q.3 A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
(a) How many balls must she select to be sure of having at least three balls of the same color?
(b) How many balls must she select to be sure of having at least three blue balls?

Q.4 What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Q.5 Atleast how many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

Q.6 How many subsets with an odd number of elements does a set with 10 elements have?

Q.7 A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
(a) are there in total?
(b) contain exactly three heads?
(c) contain at least three heads?
(d) contain the same number of heads and tails?

Q.8 In how many ways can 5 numbers be selected and arranged in ascending order from the set {1,2,3...10}?

Q.9 In how many different ways can 5 ones and 20 twos be permuted so that each one is followed by atleast 2 twos?

Q.10 How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

Q.11 How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Q.12 An agency has 10 available foster families $F_1 \ldots F_{20}$ and 6 children $C_1 \ldots C_6$ to place. In how many ways can they do this if
(a) No family can get more than one child.
(b) A family can get more than one child.

Q.13 How many distinguishable permutation can be generated from word "BANANA"?
(a) 720
(b) 60
(c) 240
(d) 120

Answers Keys

1. (Sol.)	**2.** (Sol.)	**3.** (Sol.)	**4.** (Sol.)	**5.** (Sol.)
6. (512)	**7.** (Sol.)	**8.** (252)	**9.** (Sol.)	**10.** (125)
11. (9)	**12.** (Sol.)	**13.** (b)		

Explanations

1. (Sol.)

Here repetition is allowed by default (nothing is mentioned about repetition).

Number of possible strings of 0 length (empty string) : 1

Number of possible strings of length 1 : 26

Number of possible strings of length 2 : 26×26

Number of possible strings of length 3 : $26 \times 26 \times 26$

Number of possible strings of length 3 : $26 \times 26 \times 26 \times 26$

Therefore the total number of strings of length 4 or less = $1+26 + 26^2 + 26^3 + 26^4$